Bound Lengths Based On Empirical Bayesian Scenario For A Repairable System

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***Abstract*—The present article studying the properties of Bayes prediction bound lengths of a repairable model under empirical Bayesian approach on progressive censored data. The prediction length of bounds has been obtained under the assumption that, the repair failure rate increases monotonically as time parameter increases.**

***Index Terms*—Prediction Bounds Lengths, Empirical Bayesian Approach, Repairable Model, Progressive Censored Data.**

***MSC 2010 Codes* – 62A15, 62F15, 65C05.**

I. Introduction

THE censoring arises when exact lifetimes are only partially known and it is much useful in life testing experiments for time and cost restrictions. The progressive censoring appears to be a great importance in planned duration experiments in reliability studies. In many industrial experiments involving lifetimes of machines or units, experiments have to be terminated early and the number of failures must be limited for various reasons.

The planning of experiments with the aim of reducing total duration of experiment or number of failures leads naturally to Type-I & Type-II censoring scheme. The main disadvantage of above censoring schemes is that they do not allow removal of units at points other than the termination point of an experiment. For such lifetime studies Progressively Type-II censored sampling is an important method. Live unites removed early on, can be readily used in others test, thereby saving cost to experimenter and a compromise can be achieved between time consumption and the observation of some extreme values.

In many applications, technical systems or sub-systems have $k-out-of-n$ structure, which has investigated extensively in the literature. For such a system, the system consisting of n components or subsystems, of which only $k\ (\leq n > 0)$ need to be functioning. The $k-out-of-n$ model is commonly used model in reliability theory. The system includes multi-display system in cockpits, the multi-engine system in an airplane, and the multipurpose system in a hydraulic control system. In this model, the failure of any component of the system does not influence the components still at work.

The system $(n-1)-out-of-n : G$ is consists with n components and works if and only if $(n-1)$ components among the n work simultaneously. System and each of its components can in only one of two states: working or failed. When a component fails, it kept under the repair and the other components stay in the working state with adjusted rates of failure. After repairing, a component works as new and its actual lifetime is the same as initially. If the failed component is repair before another component fails, the $(n-1)$ components recover their initial lifetime. The lifetime and time of repair are independent.

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Prakash & Kumar [6] studied the behavior of Bayes prediction length of interval under Two-Sample Bayes prediction scenario based on a repairable system recently. The right item censoring criterion is the first time introduced by them in repairable model. The present article extends the work of Prakash & Kumar [6] by introducing the Progressive censoring criterion in such repairable model.

The objective of present article is to predict the nature of future behavior of an observation when sufficient information about past and the present behavior of an event or an observation is known or given. In present paper an empirical Bayesian statistical analysis is used for predicting the future ordered statistic from considered repairable model. Under progressive ordered data, One-Sample Bayes prediction scenario is considered for studying the properties of the bound lengths.

Gherda & Boushaba [1] analyzed a repairable system with failure and repair times arbitrarily distributed. Soliman et al. [7] presented some Bayesian inference and prediction of Burr Type-XII distribution under progressive first failure censored sampling. Some Bayesian and frequentist prediction under progressive censoring was discussed by Soliman et al. [8]. Mahmoud et al. [2] studied about the Bayesian inference and prediction of generalized Pareto distribution under progressive first-failure censored data. Mohie El-Din et al. [3] presented statistical inference and prediction for the inverse Weibull distribution based on record data. Some Bayes prediction bounds lengths for right ordered Pareto Type-II data was obtained by Prakash [4]. Recently, Prakash [5], presented some Bayes estimation under progressively censored Rayleigh data. Few most recent studies are discussed above. However, a great deal of literature is available on predictive inference of the future failure distribution under progressive censoring.

II. Description Of The Model Under Study

The considered repairable model by Prakash & Kumar [6] is based on following assumptions:

The system consists with n units and having a repair facility. Initially one unit starts operating and the remaining $(n-1)$ units are kept as inactive standbys. As soon as a unit fails, it goes to repair and a standby unit is put on the operation. The repair policy is based on First Come First

Serves (FCFS), it is always open, and the repairs are perfect with negligible switch over time.

The failure time distribution of online units and repair time distribution of units under repair are assume general, independent of each other, and both are increasing failure rate (IFR) distributions. The state of the system is defined by the number of non-operative units in the system at time $t(>0)$. Further, the state n is called the down state of the system.

The system is observed under an inspection policy where inspection is made at the completion of a repair, if it starts at the beginning of a repair. This leads us to a situation where separate observations on the units performance and on repair facility are not feasible. Thus, available records are the number of failures that occurred in the time interval between two repair epochs i.e., the time instant at which a repair completes.

Following Prakash & Kumar [6], the failure rate $\rho(t); t>0$ for such repairable system is given as follows:

$$\rho(t) = \theta t^{\delta-1}; \theta>0, t>0, \delta>1. \tag{1}$$

Let us suppose an experiment in which n independent and identical units $X_1, X_2, ..., X_n$ are placed on a life test at the beginning time and first $m; (1 \leq m \leq n)$ failure times are observed. At the time of each failure occurring prior to the termination point, one or more surviving units are removed from the test. The experiment is terminated at time of the m^{th} failure, and all remaining surviving units are removed from the test.

Let $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}$ are the lifetimes of completely observed units to fail and $R_1, R_2, ..., R_m; (m \leq n)$ are the numbers of units withdrawn at these failure times respectively. Here, $R_1, R_2, ..., R_m; (m \leq n)$ all are predefined integers follows the relation (see Prakash [5] for details)

$$\sum_{j=1}^{m} R_j = n - m.$$

The joint probability density function of order statistics based on progressively Type-II censoring scheme is defined as

$$f_{X_{1:m:n}, X_{2:m:n},, X_{m:m:n}} \left(\underline{x}|\theta\right) = C_m \prod_{i=1}^{m} \cdot f\left(x_{(i)};\theta\right)\left(\frac{f\left(x_{(i)};\theta\right)}{\rho\left(x_{(i)}\right)}\right)^{R_i} \tag{2}$$

where $f(\cdot)$ be the probability density function of considered model and $\rho(\cdot)$ be the failure rate of the considered model. The progressive normalizing constant C_m is defined as

$$C_m = n\left(n-R_1-1\right)\left(n-R_1-R_2-2\right)... \left(n+1-\sum_{j=1}^{m-1} R_j - m\right).$$

Simplifying (2), get the joint probability density function of order statistics as

$$f_{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}} \left(\underline{x}|\theta\right) = C_m A_m\left(\underline{x};\delta\right)\theta^m \cdot exp\left(-\theta T_m\left(\underline{x};\delta\right)\right); \tag{3}$$

where $A_m\left(\underline{x};\delta\right) = \prod_{i=1}^{m} x_{(i)}^{\delta-1}$ and $T_m\left(\underline{x};\delta\right) = \frac{1}{\delta}\sum_{i=1}^{m}\left(1+R_i\right)x_{(i)}^{\delta}$.

Two-parameter Gamma distribution (Prakash & Kumar [6] is considered here as a conjugate prior for the parameter θ, having probability density function

$$\pi(\theta) \propto \theta^{\alpha-1} exp\left(-\beta\theta\right); \alpha>0, \beta>0, \theta>0. \tag{4}$$

Now, the posterior distribution is defined and obtained as

$$\pi^*\left(\theta|\underline{x}\right) = \frac{f_{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}}\left(\underline{x}|\theta\right)\cdot\pi(\theta)}{\int_\theta f_{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}}\left(\underline{x}|\theta\right)\cdot\pi(\theta)d\theta}$$

$$\propto \frac{\theta^m exp\left(-\theta T_m\left(\underline{x};\delta\right)\right)\cdot\theta^{\alpha-1}exp\left(-\beta\theta\right)}{\int_\theta \theta^m exp\left(-\theta T_m\left(\underline{x};\delta\right)\right)\cdot\theta^{\alpha-1}exp\left(-\beta\theta\right)d\theta}$$

$$\Rightarrow \pi^*\left(\theta|\underline{x}\right) = \frac{\left(T_m^*\left(\underline{x},\delta\right)\right)^{m+\alpha}}{\Gamma(m+\alpha)}\theta^{m+\alpha-1} \cdot exp\left(-\theta T_m^*\left(\underline{x};\delta\right)\right) \tag{5}$$

where $T_m^*\left(\underline{x},\delta\right) = T_m\left(\underline{x},\delta\right)+\beta$.

III. Empirical Bayesian Criterion

The method of maximum likelihood (ML) estimate and method of moments are two best techniques for estimating the hyper-parameters. Based on empirical Bayesian approach, unknown hyper-parameter β is estimated by the method of ML estimate when hyper-parameter α is considered to be known.

Under empirical Bayesian approach, we begin with Bayesian model: Since,

$$x_{(i)}|\theta \sim f\left(x;\theta\right)\ ;\ i=1,2,...,n$$

and

$$\theta|\beta,\alpha \sim \pi(\theta).$$

As all the units have identical $f(\cdot)$ distribution, therefore marginal density of x, say $f(x)$, can be obtained as

$$f(x) = \int f_{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}}\left(\underline{x}|\theta\right)\cdot\pi(\theta)d\theta$$

$$\Rightarrow f(x) = \frac{C_m A_m\left(\underline{x};\theta\right)\beta^\alpha}{\Gamma(\alpha)}\Gamma(m+\alpha)\cdot \left(T_m^*\left(\underline{x};\theta\right)\right)^{-(m+\alpha)}. \tag{6}$$

The maximum likelihood estimate of β based on $f(x)$ is given as

$$\hat{\beta}_{ML} = \frac{\alpha}{m}T_m\left(\underline{x};\delta\right). \tag{7}$$

Now, the empirical posterior distribution for the parameter θ is obtained by replacing hyper-parameter β by its ML estimate.

Hence, the empirical posterior distribution is obtained from equation (5) as

$$\pi_E^*(\theta|\underline{x}) = \frac{\left(\hat{T}_m(\underline{x};\delta)\right)^{m+\alpha}}{\Gamma(m+\alpha)}\theta^{m+\alpha-1} \cdot exp\left(-\theta\hat{T}_m(\underline{x};\delta)\right); \quad (8)$$

where $\hat{T}_m(\underline{x};\delta) = T_m(\underline{x};\delta)\left(1+\frac{\alpha}{m}\right)$

IV. Bayes Prediction Bound Lengths Under OneSample Plan

Let $x_{(1)}, x_{(2)}, ..., x_{(m)}$ be first m components of the observed ordered items from the considered repairable model of size n. If $Y = \left(y_{(1)}, y_{(2)}, ..., y_{(s)}\right)$ be the another independent ordered random sample of size s from same model of the future observations. Then the Bayes predicative density of future observation Y is denoted by $h(y|\underline{x})$ and obtained by simplifying following relation

$$h(y|\underline{x}) \propto \int_\theta f(y;\theta)\cdot\pi_E^*(\theta|\underline{x})\,d\theta;$$

$$\Rightarrow h(y|\underline{x}) = (m+\alpha)\left(\hat{T}_m(\underline{x};\delta)\right)^{m+\alpha} y^{\delta-1} \cdot\left(\hat{T}_m(\underline{x};\delta)+\frac{y^\delta}{\delta}\right)^{-m-\alpha-1}. \quad (9)$$

The Bayes predictive density function expresses the plausibility of Y given data and the prior information regarding the parameter. Now, the Bayes predictive bounds with coverage $(1-\tau)$ is defined for future observation Y as

$$Pr(l_{1E} \le Y \le l_{2E}) = 1-\tau,$$

Here l_{1E} and l_{2E} are the lower and upper Bayes prediction bounds for random variable Y, and $(1-\tau)$ is called the confidence prediction coefficient. The One-sided $100(1-\tau)\%$ Bayes prediction bounds are obtained by solving following equality

$$Pr(Y \le l_{1E}) = \frac{\tau}{2} = Pr(Y \ge l_{1E}). \quad (10)$$

Solving (10), the lower and upper empirical Bayes prediction bounds for Y are obtain as

$$l_{1E} = \left(\delta\hat{T}_m(\underline{x};\delta)\,\tau^*\right)^{\frac{1}{\delta}} \quad (11)$$

and

$$l_{2E} = \left(\delta\hat{T}_m(\underline{x};\delta)\,\tau^{**}\right)^{\frac{1}{\delta}}; \quad (12)$$

where $\tau^* = (\tau_1)^{-\lambda}-1$, $\tau^{**} = (\tau_2)^{-\lambda}-1$, $\lambda = (m+\alpha)^{-1}$, $\tau_1 = \left(1-\frac{\tau}{2}\right)$ and $\tau_2 = \left(\frac{\tau}{2}\right)$.

Similarly, the central coverage $100\tau\%$ Bayes prediction bounds for the future observation Y are obtained similarly by solving following equality

$$Pr(Y \le l_{1EC}) = \frac{1-\tau}{2} = Pr(Y \ge l_{1EC}). \quad (13)$$

Solving (13), the lower and upper empirical Bayes prediction bounds for the future random observation Y are given as

$$l_{1EC} = \left(\delta\hat{T}_m(\underline{x};\delta)\,\omega^*\right)^{\frac{1}{\delta}} \quad (14)$$

and

$$l_{2EC} = \left(\delta\hat{T}_m(\underline{x};\delta)\,\omega^{**}\right)^{\frac{1}{\delta}}; \quad (15)$$

where $\omega^* = \left(\left(\frac{1+\tau}{2}\right)^{-\lambda}-1\right)$ and $\omega^{**} = \left(\left(\frac{1-\tau}{2}\right)^{-\lambda}-1\right)$.

Hence, the empirical Bayes prediction bound lengths under One-Sided and Central coverage are obtained as

$$I_E = l_{2E} - l_{1E}$$

and

$$I_{EC} = l_{2EC} - l_{1EC}.$$

Table 1: Censoring Scheme for Different Values of m

$Case$	m	$R_i\ \forall\, i = 1, 2, ..., m$
1	05	1 2 1 0 1
2	10	1 0 0 3 0 0 1 0 0 1
3	15	1 0 2 0 0 1 0 2 0 0 1 0 0 1 2

V. Numerical Analysis

We illustrate the procedure by presenting a complete analysis under a simulated data set. The random samples are generated as follows:

Generates the values of parameter θ through prior density $\pi(\theta)$ for a given set of prior parameters $\alpha(= 0.50, 1.00, 2.50, 5.00)$ and corresponding values of prior parameter β is estimated by its ML estimate $\hat{\beta}_{ML}$.

Using above generated values of θ with $\delta(=1)$; a set of $10,000$ random samples of size $n = 20$ has been drawn from underlying model. For selected values of progressive censored sampling scheme $m(= 05, 10, 15)$ and level of significance $\tau(= 99\%, 95\%, 90\%)$; the empirical Bayes prediction lengths of bounds are obtained and presents them in Table 2. Table 1 shows the different progressive censoring scheme.

It is observed from the table that, the lengths of empirical Bayes prediction bounds for both cases tend to be wider as progressive censoring scheme m increases and changed, when other parametric values are fixed. The length of bounds expended also, when hyper-parameter α increases.

It is also seen that the length of bounds tends to be closer when the level of significance τ decreases for both cases when other parametric values are fixed.

However, it is remarkable that the central coverage empirical Bayes prediction length of bounds are wider as compare to one-sided empirical Bayes prediction length of bounds for higher significance level. For 90% significance and $\alpha \ge 2.50$, the one-sided prediction length of bounds is wider as compare to central coverage criterion.

Table 2: Empirical Bayes Prediction Bound Length of Intervals under One-Sample Technique

$n = 20$		One - Sided			Central Coverage		
m	$(\alpha) \downarrow \tau \rightarrow$	90%	95%	99%	90%	95%	99%
05	0.50	0.4971	0.5777	0.6981	1.0079	1.1854	1.3243
	1.00	0.6158	0.6278	0.7147	1.0146	1.2019	1.4715
	2.50	1.0696	1.1808	1.4087	1.0401	1.2307	1.8504
	5.00	1.3324	1.3625	1.6472	1.0575	1.3753	1.9805
10	0.50	0.6331	0.6974	0.7718	1.0094	1.1942	1.3932
	1.00	0.8079	0.8934	0.9884	1.0417	1.2513	1.5551
	2.50	1.2181	1.2537	1.6219	1.0752	1.2817	1.8622
	5.00	1.3813	1.4154	1.8306	1.1278	1.5173	1.9829
15	0.50	0.7112	0.8487	0.8729	1.0133	1.2541	1.4239
	1.00	0.8285	0.9548	1.0079	1.0933	1.2936	1.5928
	2.50	1.2275	1.4235	1.714	1.1196	1.5437	1.8799
	5.00	1.4075	1.6519	1.8889	1.2565	1.6853	1.9894

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